

The internal structure of late-type main-sequence stars

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Abstract

Homology scaling laws for main-sequence stars are derived, and used to estimate how late-type stars evolve during their core hydrogen-burning phase. Though not exactly representative of realistic stellar models, the scaling laws do provide a useful method of making estimates of small perturbations either to the initial conditions or to the physics used in the so-called standard theory of stellar evolution. In particular, evolution with varying gravitational constant and varying mass are considered explicitly. The scaling laws are used to determine how gross observable parameters such as luminosity, neutrino flux or acoustic oscillation frequencies depend on the mass, composition and age of the star. By inverting the relations it is shown that with the precision of the best measurements of luminosity, effective temperature and the heavy-element to hydrogen abundance ratio, supplemented with a knowledge of the principle parameters characterizing the high-order acoustic oscillation spectrum, theoretical models could be calibrated to determine mass and age to within about 20 per cent. No useful information about the helium abundance can be obtained in this way.

Introduction

Much of Bengt Strömngren's work was concerned with or motivated by the desire to understand the nature of stars. In his early days he invested a substantial effort in modelling the internal structure, as also did Eddington and Milne at that time. The critical dependence of stellar structure on molecular weight and opacity implied that the position of a star on the Hertzsprung-Russell diagram depended not only on its mass but also on chemical composition, and therefore potentially afforded a means of inferring the proportion of hydrogen to heavy elements in stars. In 1937 the work by von Weizsäcker on nuclear transmutation made it evident that helium was the principal product of thermonuclear reactions in stars, and immediately Strömngren considered the implications of helium being a major constituent of stellar material. Now there was an additional important parameter to determine, and the problem of inferring the nature of a star from observation became richer, and correspondingly more difficult. The central question was then: What are the relative abundances X , Y and Z of hydrogen, helium and heavy elements in the interior of a star? And subsequently it was asked: Can we distinguish observationally between stars of different ages? These are amongst the questions I shall be addressing again in this lecture.

Most of the early work on stellar structure assumed that chemical composition was

uniform throughout the interior of a star. This permitted the extensive use of homology scaling laws to compare the properties of one star with another, at least on the main sequence where issues of dense highly degenerate cores and extended giant envelopes do not arise. The sun occupied a central position in the investigations, providing an accurately determined standard with which to calibrate theory. Subsequently it became evident that stars are not chemically homogeneous, though it was not until the advent of electronic computers that it was possible to calculate, or perhaps I should say estimate, the material distribution in evolved stars. Now it is possible to carry out quite detailed numerical computations, and compare the outcome with a wealth of observational data.

So where does that leave the methods used by Strömgren and his contemporaries in those pioneering days? They are not outdated. Even though computers have taken over the role of constructing detailed models, simple scaling arguments, even under circumstances in which the conditions that justify them are not strictly satisfied, are extremely important aids to rationalizing the results of numerical computations, and so to increasing our understanding of the complicated balance of processes that determine stellar structure. By representing the results in rough analytical terms, people like me who are unskilled at interpreting vast arrays of precise numerical computer output can appreciate more readily what are likely to be the most important factors determining the observable properties of stars. In this lecture I shall illustrate this by discussing what is perhaps the most basic aspect of the subject, namely the hydrogen-burning main-sequence phase. I shall keep the discussion as simple as possible, ignoring unnecessary complications without justifying why they are unnecessary: unlike the pioneers in the days before electronic computing, I can always consult numerical solutions of more complicated and hopefully more realistic theoretical models to be reassured that my approximations do not distort the picture too severely.

One might well ask what the purpose of such an exercise is. Surely the main sequence is so well understood that there can be hardly any more to say at so elementary a level. To be sure we have the solar neutrino problem, but after so many thousands of hours of computer time have been dedicated to the unsuccessful search for but one theoretical model of the sun that is not in conflict with observation, rough analytical estimates can hardly be of any real value. That is certainly the view held by many workers in the field. However, I do not support it for the following reasons. First, by thinking in very simple terms one is forced to step back from the morass of detail that is present in the modern computer programmes, and perhaps then one can see more clearly what might be deficient in the theory. Second, with a simple picture in mind one can predict the results of new computations; this is important because it is extremely useful to know the answer to a problem in advance when trying to judge whether the inevitable errors that creep into new modifications to a computer programme have been eradicated. Finally, and most important of all, when one can

really find no errors remaining and when the numerical results persist in disagreeing with expectation, one is forced to modify one's simple picture: that is what constitutes real learning.

I am not claiming that one can necessarily use simple arguments to make precise absolute comparisons with observation. One needs an accurately computed detailed model for that. But what one can do is to enquire how that model is modified when certain parameters are changed, or when certain physical phenomena are modified or even introduced into consideration, provided the modification is not too large. One is essentially carrying out approximate perturbation theory.

I must point out also that there is a new reason for rediscussing main-sequence evolution: it is provided by the body of new seismic data that have recently been gathered from solar observations, and some similar data that we anticipate will be obtained in the near future from other stars. These data provide additional and different constraints from those imposed by the bulk parameters that have been obtained by classical astronomical techniques, such as mass and position on the Hertzsprung-Russell diagram. In confronting theory with them it is again productive initially to think in very simple terms.

2. *Simple main-sequence evolution*

What I mean by 'simple' evolution is the theoretical study of spherically symmetrical stellar models whose temporal variation on the main sequence (possibly after an initial transient associated with the approach to the main sequence) is determined solely by the gradual nuclear transmutation of hydrogen into helium; in this description the star's mass is constant, and there is no transport of chemical species through the star except in convection zones.

The subject has been studied extensively. In particular, the theory has been applied to the mass-luminosity relation and the position of stars on the Hertzsprung-Russell diagram. The sensitivity of the results to chemical composition, and to uncertainties in the theoretical description of energy transport in the convection zones, has also been investigated.

As a result of the discrepancy between the observed and theoretical values of the solar neutrino flux, the theory has been quite highly refined. The microphysics especially has been reassessed, to provide a more secure basis for the procedures by which the nuclear reaction rates, the equation of state and the opacity are determined. The sensitivity of solar models computed in this simple way to the obvious uncertainties in the microphysics has been extensively investigated, and summarized recently in two important papers by Bahcall *et al.* (1982) and Bahcall and Ulrich (1988), which provide a useful basis for comparing other possibly more realistic models. Indeed, solar models computed in this simple way are now commonly called 'standard', even

though the details are continually being modified. In the attempt to resolve the neutrino problem, uncertain parameters have understandably sometimes been set to extremes of plausibility.

Before proceeding into any detail it is useful to list some of the more obvious features of the so-called standard models:

- (i) hydrostatic balance and thermal balance,
- (ii) no rotation nor dynamically significant magnetic field, and hence spherical symmetry,
- (iii) no accretion nor mass loss,
- (iv) no macroscopic meridional motion, other than small-scale turbulence in the unstably stratified convection zone; therefore, in particular, no mixing of entropy from convection zones into radiative regions, and no mixing of the products of nuclear reactions in lower main-sequence stars that do not have convective cores, and therefore
- (v) no wave transport of energy or momentum.

It is also assumed that the generally accepted laws of physics are valid. Thus, for example, with respect to appropriate units of mass, length and time in which Planck's constant h and the speed of light c are constant:

- (vi) G is constant,

where G is the gravitational constant.

These features, which are written into the theory, are essentially assumptions, though they are not wholly unjustified. We note that most main-sequence stars do not appear to vary substantially on a dynamical timescale, and therefore that hydrostatic balance must be a very good first approximation. Moreover, since for all but perhaps the most massive stars the characteristic nuclear transmutation time substantially exceeds the thermal diffusion time, most main-sequence stars have had time to achieve thermal balance. Therefore they are presumably in thermal balance, unless some instability has recently upset it. Studies of thermal stability, particularly of the sun, generally provide little cause for doubt. Although the sun is observed to rotate (and spectrum line-width measurements of other stars suggest that the solar rotation is not grossly atypical of stars in its spectral class) the centrifugal force is extremely small compared with gravity. This is consistent with the figure of the sun having been observed to differ from being spherical by no more than about 1 part in 10^6 . Early-type stars rotate more than 100 times faster, but except in extreme cases the neglect of centrifugal force in the hydrostatic equation is probably not a serious flaw in the models. For most main-sequence stars there is little evidence of substantial accretion or mass loss.

It is more difficult to justify the remaining assumptions. Although centrifugal force is unimportant to the hydrostatic balance in the radial direction, it is potentially important horizontally. Indeed, in a uniformly rotating star (or a star in which angular velocity Ω is instantaneously a function only of distance from the rotation

axis) the rotational terms in the momentum equation can be represented as the gradient of a potential which can be added to the gravitational potential, thereby changing the meaning of horizontal; von Zeipel (1924) pointed out that except under very contrived circumstances the pressure cannot be made constant on surfaces of constant total potential, which Eddington (1925) and Vogt (1925) realised would lead to possibly significant circulatory motion. Basically, the reason is that thermal diffusion tends to make surfaces of constant temperature, and consequently surfaces of constant pressure, more nearly spherical than what is required for hydrostatic balance. If Ω is not a function of distance from the axis alone, the advection terms cannot even be derived from a potential; then they can never be balanced by a pressure gradient however contrived, and motion must necessarily ensue.

This conclusion might be invalidated if Lorentz forces associated with an internal magnetic field were taken into account. A general study of rotating magnetic equilibrium configurations of realistic stellar models has not been undertaken, but it seems likely that any that might exist would be unstable (Pitts and Tayler, 1985).

The first serious attempt to calculate the flow resulting from rotational imbalance in an isolated star was carried out by Sweet (1950). In his calculation, as in most that have followed, Ω was assumed known, and the influence upon it of advection of angular momentum by the circulation was not taken into account. A self-consistent steady solution of the equations governing a rotating nondegenerate nonmagnetic star has never been found; any that might exist is likely to be unstable (*e.g.* Gough, 1976; Tassoul, 1978; Zahn, 1989). Nevertheless, it appears that the characteristic circulation time is likely to be of order of the thermal diffusion time multiplied by the ratio of the gravitational to the centrifugal acceleration. It is called the Eddington-Sweet time, and for sun-like stars is about 100 times the characteristic nuclear evolution time, assuming the surface angular velocity to be characteristic of the interior rotation.

I appear to have digressed quite a long way from my simple picture of a star, and I have done so quite deliberately in order to draw attention to one of its possible deficiencies: although a rotationally driven circulation is too slow to be dynamically important, it could have a marked effect on the chemical evolution of the star by transporting the products of the nuclear reactions away from the site of their creation. An Eddington-Sweet time 100 times the nuclear time may at first seem too long to be significant for the structure of the sun, but a little thought makes one realise that that might not be so. In a subject with great uncertainty it is not difficult to erode confidence in a factor of 100, particularly when it is appreciated that the circulation rate is proportional to the square of the angular velocity. First, we know that the sun was rotating more rapidly in the past than it is now (the solar wind today is removing angular momentum on a timescale comparable with, though apparently somewhat greater than the age of the sun). This is consistent with the observation that the rotation rates of stars in young clusters, notably the Hyades and the Pleiades, are

substantially greater than those of older but otherwise similar stars. So perhaps rotationally induced material mixing has significant consequences early in main-sequence evolution. Secondly, the scant seismological evidence that concerns the solar core indicates that even today the core might be rotating perhaps three times faster than the surface (Duvall *et al.*, 1984; Gough, 1985) though little confidence can yet be given to what at present is no more than a slight hint. Thirdly, we know that steep gradients of angular velocity can enhance the circulation rate, and a relatively rapid rotation of only the core means that substantial gradients might exist. And finally, the conclusion that a steady rotating star cannot be stable implies that the motion must actually vary with time. On what time scale we do not know, though one is tempted to ponder over the stellar cycle time, 22 years in the case of the sun, as a candidate. What are the consequences?

Transport of momentum and energy by waves is commonly thought to be negligible. Most nonadiabatic linear studies of g modes excited by nuclear reactions in the core of the sun have found instability at some epoch on the main sequence. One might anticipate that these modes would have the capacity to redistribute not only momentum and energy, but also the helium produced by the nuclear reactions. However, it must be appreciated that the uncertain interaction between the modes and other forms of motion, such as convection, leaves considerable room for doubt. So perhaps in reality all the modes are stable. Moreover, Dziembowski (1983) has argued that even if the modes were excited, their nonlinear development would be so severely curtailed by resonant coupling to stable modes that their ability to induce substantial transport of material in the core would be negligible. However, it is not wholly out of the question that their influence on the distribution of angular momentum throughout the star is not insignificant.

Notwithstanding this list of concerns, it is very likely that the broad picture provided by the simple models is basically correct. Therefore without doubt it is extremely useful to study this picture, provided a healthy scepticism of the fine details is maintained.

3. *Simplified equations of stellar structure*

Assuming the star to be static and spherically symmetric, the equations of stellar structure may be written,

$$\frac{dp}{dr} = -\frac{Gm\rho}{r^2}, \quad (3.1)$$

$$\frac{dm}{dr} = 4\pi r^2 \rho, \quad (3.2)$$

$$\frac{dT}{dr} = -F, \quad (3.3)$$

$$\frac{dL}{dr} = 4\pi r^2 \rho \varepsilon, \quad (3.4)$$

where r is a radial coordinate, p , ρ , and T are pressure, density, temperature, and the intermediate variable m is the mass enclosed within the sphere $r = \text{constant}$; in addition $L(r)$ is the internal luminosity (the integral of the energy flux over the surface of the sphere $r = \text{constant}$) and ε is the rate of generation of thermal energy per unit mass by nuclear reactions. If the star were slowly expanding or contracting, on a timescale much greater than the characteristic dynamical timescale of the star, the hydrostatic equation (3.1) would remain valid but a work term would need to be added to the energy conservation equation (3.4). The function F in the energy transport equation (3.3) depends on whether the radius at which it is defined is in a radiative or a convective region. In an optically dense radiative region

$$F = \frac{3\kappa\rho L}{16\pi r^2 a c T^3}, \quad (3.5)$$

where κ is the Rosseland mean opacity, a is the radiation density constant and c is the speed of light. In convective cores and throughout most of the convective envelopes of dwarf stars the stratification is essentially adiabatic, and

$$F \simeq -T \left(\frac{\partial \ln T}{\partial \ln p} \right)_s \frac{d \ln p}{dr} = \frac{Gm\rho T}{r^2 p} \left(\frac{\partial \ln T}{\partial \ln p} \right)_s, \quad (3.6)$$

the partial thermodynamic derivative being taken at constant specific entropy s .

Equation (3.6) is invalid in the nonadiabatic boundary layers at the edges of convection zones. The boundary layer most significant to stellar structure is probably that at the top of convective envelopes, in and immediately beneath the photospheric layers. For my limited purposes I shall not need to know the detailed stratification of those boundary layers, so I shall not need to discuss the prescriptions that are employed to calculate it.

Notice that the Reynolds stress associated with the convective motion has been omitted from the hydrostatic equation (3.1). That stress is important only in the subphotospheric superadiabatic boundary layers of convective envelopes, whose details I have already decided to ignore.

Finally I must add an equation of state. The perfect-gas law is an adequate approximation for my purposes:

$$p = \frac{\mathfrak{R}\rho T}{\mu}, \quad (3.7)$$

where \mathfrak{R} is the gas constant and μ is the so-called mean molecular weight of the stellar material; I ignore degeneracy and radiation pressure: These are small in sun-like stars, and I wish to keep the discussion simple. If all species were completely ionized

$$\mu^{-1} = 2X + \frac{3}{4}Y + \frac{1}{2}Z, \quad (3.8)$$

where X , Y and Z are the relative abundances by mass of hydrogen, helium and heavy elements respectively.

4. *Stellar scaling laws*

Because the right-hand sides of Equations (3.1)-(3.7) are all products or quotients of variables, one can seek scaling laws that preserve the functional form of the solution. That would require also that ε , κ and μ can also be similarly expressed; strictly speaking that is not the case, but one can make progress with power-law approximations provided the range of variation of conditions is not too great. In particular, I set

$$\varepsilon = \varepsilon_0 \rho T^\eta \quad (4.1)$$

$$\kappa = \kappa_0 \rho^\lambda T^{-\nu} \quad (4.2)$$

$$\mu = \mu_0 \quad (4.3)$$

where $\varepsilon_0, \kappa_0, \mu_0$ are functions of X, Y and Z .

One now seeks homologous transformations of a solution to the structure equations. For any dependent variable q , say, one sets

$$q(r) = Q\tilde{q}(x), \quad (4.4)$$

where $x = r/R, R$ being the radius of the star, and demands that the function $\tilde{q}(x)$ is independent of the scaling factor Q . Eq. (3.2) implies that ϱ scales as M/R^3 . Then Equations (3.1) and (3.3)-(3.7) require the scaling factors to satisfy the relations

$$PR^{-1} \propto GMR^{-2} (M/R^3), \quad (4.5)$$

$$TR^{-1} \propto F, \quad (4.6)$$

$$LR^{-1} \propto \varepsilon_0 R^2 (M/R^3)^2 T^\eta, \quad (4.7)$$

where

$$F \propto \kappa_0 R^{-2} (M/R^3)^{1+\lambda} T^{-(3+\nu)} L \quad (4.8)$$

in a radiative star, or

$$F \propto \mu_0 GMR^{-2} \quad (4.9)$$

in a convective star, and

$$P \propto \mu_0^{-1} (M/R^3) T. \quad (4.10)$$

In a fully convective star the hydrostatic scaling apparently decouples from the energy scaling; relations (4.6) and (4.9) combine to reproduce relation (4.10) –

indeed, relation (4.9) was derived essentially by requiring that to be so – leaving the system (4.5)-(4.7), (4.9) and (4.10) incomplete. The system is closed by considering the radiative balance in the photospheric regions, thereby relating the energy flux F to the physical state of the atmosphere (e.g. Hayashi, Hōshi and Sugimoto, 1962). Note that the scaling factors M and L apply to the luminosity and mass variables for any fixed value of x , and in particular for $x = 1$. They are, therefore, measures of the total mass and luminosity of the star.

I should point out here that few stars are convective throughout, and no stars are everywhere in radiative equilibrium. Therefore these scaling laws cannot hold exactly, even if the simple approximations (4.1)-(4.3) were exact. Nevertheless, if a star were dominated by either a radiative or a convective zone, one might hope that the scaling laws were roughly valid, at least for that zone. It is with this hope, though in recognition of possible pitfalls, that my discussion proceeds.

Equations (4.5)-(4.10) can be solved to determine how R and L depend upon M and the coefficients μ_0 , ε_0 and κ_0 . Except for very low-mass stars, most of the structure is determined by the radiative equilibrium condition (4.8). Then

$$R \propto (\mu_0 G)^{(\eta-\nu-4)k} (\varepsilon_0 \kappa_0)^k M^{(\eta-\nu+\lambda-1)k}, \quad (4.11)$$

where

$$k = (\eta - \nu + 3\lambda + 3)^{-1}, \quad (4.12)$$

and

$$L \propto (\mu_0 G)^a \varepsilon_0^{-b} \kappa_0^{-(1+b)} M^e, \quad (4.13)$$

where the exponents a , b and e (a no longer being the radiation constant) are given by

$$\begin{aligned} a &= \eta - (\eta + 3)(\eta - \nu - 4)k \\ b &= (\eta + 3)k - 1 \\ e &= \eta + 2 - (\eta + 3)(\eta - \nu + \lambda - 1)k. \end{aligned} \quad (4.14)$$

It is useful to rewrite the scaling laws in terms of chemical composition. To this end we note that $\epsilon_0 \propto X^2$ with $\eta \simeq 4$ for the proton-proton chain, and recall Equations (4.3) and (3.8) determining μ_0 . For stars somewhat more massive than the sun, the CNO cycle dominates the thermonuclear energy production and $\epsilon_0 \propto XZ$ with $\eta \simeq 16$. If opacity is dominated by bound-free or free-free transitions, Kramers' opacity law holds: $\lambda = 1$ and $\nu = 3.5$. When bound-free transitions dominate (as for young Population I stars), $\kappa_0 \propto (1+X)Z$, whereas when free-free transitions dominate (as for extreme Population II stars), then $\kappa_0 \propto 1+X$. The sun is between these two extremes; for sun-like stars I shall adopt the approximation $\kappa_0 \propto (1+X)Z^d$, with $d = 0.5$. [In very massive stars where electron scattering dominates, $\kappa_0 \propto 1+X$, $\lambda = 0$ and $\nu = 0$.] Provided that only a limited range of X and Z are considered, these formulae can be approximated by power laws. Thus, Equation (4.13), for example, can be rewritten

$$L \propto G^a X^{-f} Z^{-g} M^e, \quad (4.15)$$

where

$$f \simeq \frac{5aX_0}{3 + 5X_0} + \frac{2b + (3b + 1)X_0}{1 + X_0}, \quad (4.16)$$

X_0 being a typical value of X , and $g = (1+b)d$, which varies between 0 and $1+b$ depending on the processes that dominate the opacity. In deriving Equation (4.16), which was carried out simply by equating the logarithmic derivatives of L given by Equations (4.13) and (4.15), Z was ignored in the expression (3.8) for μ_0 .

As an example I evaluate the expressions for L for sun-like stars, taking $\lambda = 1$, $\nu = 3.5$, $\eta = 4$ and $X_0 \simeq 0.7$. Equations (4.13) and (4.15) then become

$$L \propto (\mu_0 G)^{7.8} \epsilon_0^{-0.08} \kappa_0^{-1.1} M^{5.5} \propto G^{7.8} X^{-4.8} Z^{-0.55} M^{5.5}. \quad (4.17)$$

The corresponding result for somewhat more massive stars powered by the CNO cycle ($\eta \simeq 16$) but with the same composition and opacity law is

$$L \propto (\mu_0 G)^{7.3} \epsilon_0^{-0.03} \kappa_0^{-1.0} M^{5.2} \propto G^{7.3} X^{-5.8} Z^{-0.5} M^{5.2}. \quad (4.18)$$

Some care should be exercised in interpreting Equation (4.11). For upper main sequence stars the formula is roughly correct, with R being interpreted as the radius of the star. But lower main-sequence stars have extensive convective envelopes, which cause them to deviate from a homologous sequence. For a sun-like star the convective envelope, though extending over a substantial fraction of the radius (about 30 per cent in the case of the sun), has very little mass (and therefore very little weight), and does not have a serious influence on the radiative interior. Equation (4.11) might therefore be used as a rough guide for determining the characteristic scale of variation of the radiative interior, and hence Equation (4.13), which depends on that scaling, remains approximately valid. However, the convective envelope causes the radius of the star to be less than what it would have been had convection not been operative, and the increasing relative extent of the convective envelope as M decreases therefore implies that the actual radius R increases more rapidly with M than is suggested by Equation (4.11). The dependence on X and Z is modified too, because the extent of the convection zone is determined partly by opacity. This behaviour is strictly nonhomologous: to obtain analytical estimates of the stellar radius requires a considerably more sophisticated discussion than that which I am attempting here. In practice stellar models are commonly computed using a relation between heat flux and temperature gradient in the convection zone based on a mixing-length formalism, and the resulting stellar radius depends not only on M and the quantities $\mu_0 G$, ϵ_0 and κ_0 , but also on a parameter α , the ratio of mixing length to pressure scale height, which is usually regarded as a constant. Thus I write

$$R \propto X^h Z^j \alpha^u M^v. \quad (4.19)$$

For sun-like stars $h \simeq -1$, $j \simeq -0.5$, $u \simeq -0.2$ and $v \simeq 1.2$. The exponent v declines to about unity as M decreases below M_\odot . For high-mass stars $v \simeq 0.75$. The transition between the two extremes of mass occurs for stars of about a solar mass and somewhat higher, and is determined partly by the diminution of the extent of the convective envelope as M increases, and partly by the transition from the p-p chain to the CNO cycle. As I pointed out above, the luminosity L is quite insensitive to the structure of the convective envelopes of sun-like stars, so I add no dependence on α to the scaling (4.17).

Finally, I point out that the slope of the main sequence in the Hertzsprung-Russell diagram (actually the $\log L - \log T_e$ diagram) can be obtained from the scalings (4.15) and (4.19) and the black-body radiation law:

$$L = 4\pi R^2 \sigma T_e^4, \quad (4.20)$$

where σ is the Stefan-Boltzmann constant, and from here on L will be regarded as the (surface) luminosity $L(R)$ of the star.

5. *Main-sequence evolution*

The transmutation of hydrogen into helium causes X to decrease. Thus, μ_0 increases, and ϵ_0 and κ_0 decrease, causing L , according to the scaling (4.13), to rise. The major influence is through the variation of μ_0 , whose increase demands a higher core temperature to produce pressure enough to support the weight of the star above. The evolution of the star is not homologous, however, because the nuclear reactions take place only in the inner core, where the temperature is high. Lower main-sequence stars have radiative cores, and the products of the nuclear reactions remain *in situ*; in upper main-sequence stars the core is convective, which homogenizes its chemical composition, but the reaction products do not mix into the radiative envelope. Thus in either case the increase in μ_0 occurs only in the innermost regions of the star. Nevertheless, it is instructive first to assume the star to follow the homologous scaling law (4.13), and afterwards to consider the errors introduced by that assumption. Such an analysis was carried out by Strömberg (1952) for upper main-sequence stars, which at the time were believed to be fully mixed by rotationally induced (Eddington-Sweet) circulation currents.

The rate of change of the hydrogen abundance is given by

$$-ME \frac{dX}{dt} = L, \quad (5.1)$$

where $E \approx 0.007c^2$ is the energy released per unit mass in converting hydrogen to helium. When coupled with the relation (4.13) and the appropriate expressions for μ_0 , ϵ_0 and κ_0 , this determines how X and L vary with time t . The formula for κ_0 may be evaluated at constant Z , since the effect of nuclear reactions on opacity has only a small influence on the structure of the star. The resulting equation is rather cumbersome to solve, but it can be simplified substantially if the scaling law (4.13) is replaced by the exponential approximation

$$L = L_0 \exp [-\beta(X - X_0)], \quad (5.2)$$

where X_0 and $L_0 = L(X_0)$ are constants whose values are characteristic of X and

L [and which in practice I shall take to be the initial values of X and L], and β is a constant of order unity which is determined, like f , by equating the logarithmic derivatives of expressions (4.13) and (5.2) at the initial value X_0 of X . Thus, for example, taking $\kappa_0 \propto 1+X$ (at constant Z) and assuming the p-p chain to dominate the nuclear reactions,

$$\beta \simeq X_0^{-1} f = \frac{5a}{3+5X_0} + \frac{2b}{X_0} + \frac{1+b}{1+X_0}. \quad (5.3)$$

Using the values of a and b calculated for formular (4.17), with $X_0 \simeq 0.7$, which are characteristic of sun-like stars, yields $\beta \simeq 6.8$.

The integration of Equations (5.1) and (5.2) is straightforward, yielding

$$X = X_0 + \beta^{-1} \ln(1 - t/\tau_n), \quad (5.4)$$

$$L = L_0 (1 - t/\tau_n)^{-1}, \quad (5.5)$$

where τ_n is a characteristic nuclear timescale:

$$\tau_n = \frac{EM}{\beta L_0}, \quad (5.6)$$

and I have chosen the constant of integration such that $X = X_0$, $L = L_0$ at $t = 0$. Note that according to Equation (5.4), the time t_n at which hydrogen is exhausted is given by

$$t_n = \tau_n (1 - e^{-\beta X_0}). \quad (5.7)$$

In practice the evolution is not homologous. Indeed, the inner regions of the star contract, as the scaling (4.11) suggests, whereas, according to the scaling (4.19), the outer regions of a sun-like star expand. Nuclear reactions modify the composition in only the central regions of the star, and consequently in those regions X varies

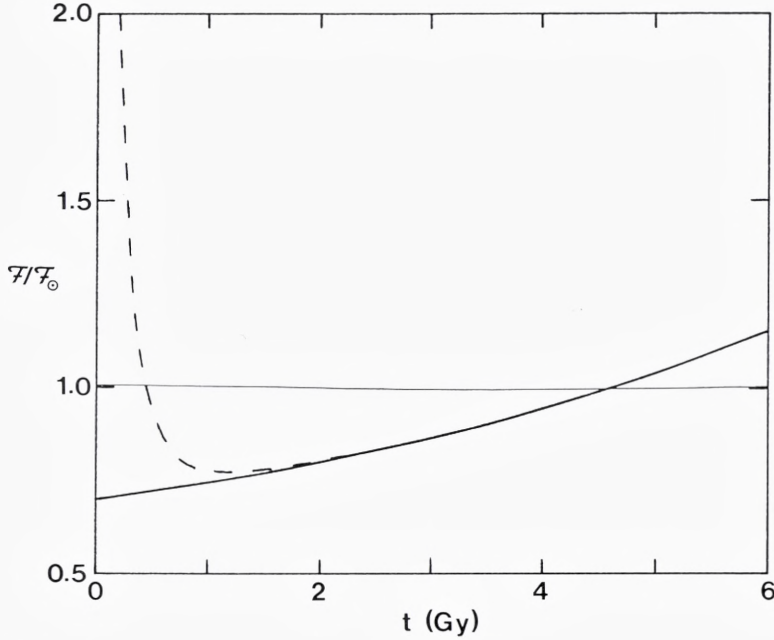


Fig. 1. The thick continuous curve illustrates the evolution of the solar constant $\mathcal{F}(t) \propto L$ according to Equation (5.8), normalized to the present value \mathcal{F}_\odot . The thin (almost horizontal) continuous curve shows the evolution of $\mathcal{F}/\mathcal{F}_\odot$ with L/L_\odot given by Equation (8.4) when the gravitational constant declines according to Equation (8.1), with $q = 1$ and $\tau_c(T) = 1.2 \times 10^{11}$ y. (A very similar result is obtained with $q = 0.1$ and $\tau_c = 1.3 \times 10^{11}$ y.) The dashed curve is $\mathcal{F}/\mathcal{F}_\odot = L/L_\odot$, given by Equations (9.3) and (9.4), for a model losing mass according to Equation (9.1) with $t_0 = 0.05t_\odot$ and $\Lambda = 0.45$. The initial luminosity is $L_0 = 5.8L_\odot$. The two models with varying G or M both satisfy $\dot{L}(t_\odot) = L_\odot$.

substantially more rapidly than predicted by Equation (5.4). L varies more rapidly than predicted by Equation (5.5), but the discrepancy is not as great as for X . Indeed, in the case of the sun Equation (5.5) fits numerical computations very well if τ_n is replaced by $\tau_\odot = EM/\beta L_\odot$, yielding

$$L \simeq \frac{L_0}{1 - 0.3t/t_\odot} \simeq \frac{L_\odot}{1 + 0.4(1 - t/t_\odot)}, \quad (5.8)$$

where t_\odot and L_\odot are the current age and luminosity of the sun. The right-hand side of Equation (5.4) then estimates the variation of the mean hydrogen abundance of the

star. Equation (5.7) can no longer be used to estimate accurately the main-sequence lifetime, however, because that is determined predominantly by conditions in the core, though it is likely that τ_{\odot} scales in roughly the same way as τ_n . According to Equation (5.8), $L(t)$ increases monotonically with t from its initial value $L_0 \simeq 0.7L_{\odot}$. Thus $\tau_n \simeq 1.3\tau_{\odot}$. The variation is illustrated in Figure 1.

6. Calibration of solar models

Equation (5.8) describing the temporal variation of the solar luminosity L has been calibrated such as to give the observed luminosity L_{\odot} at the current age t_{\odot} . When computing theoretical models, that calibration is carried out by adjusting the initial composition. Usually Z/X_0 is specified and then X_0 is adjusted to give the correct luminosity. As mentioned above, in practice it is necessary also to adjust the mixing-length parameter α in order to obtain the correct radius R_{\odot} , but since changing α hardly influences the structure of the radiative interior I ignore it here.

To estimate how X_0 , or equivalently the initial helium abundance Y_0 , must be adjusted, one can use the homology scaling law (4.17) keeping L and M (and, of course, G) constant. Once again taking $X_0 = 0.7$, one thus obtains

$$\frac{\partial \ln Y_0}{\partial \ln (Z/X_0)} = -\frac{X_0}{1 - X_0} \frac{\partial \ln X_0}{\partial \ln (Z/X_0)} \simeq 0.30. \quad (6.1)$$

This is the same as the value computed numerically by Bahcall *et al.* (1982) from standard solar models.

One can estimate how $L(t)$ is affected by changes in chemical composition using the arguments of the previous section. First, if one assumes evolution to be homologous, then the only modification arises from the dependence of β on X_0 . In particular it is straightforward to demonstrate from the relations (4.17) and (5.3)-(5.6) that the initial luminosity L_0 satisfies

$$\frac{\partial \ln L_0}{\partial \ln \beta} \simeq -0.3, \quad \frac{\partial \ln L_0}{\partial \ln (Z/X_0)} \simeq 0.02. \quad (6.2)$$

In reality, however, the evolution is not homologous, and the star evolves more

rapidly than is predicted by Equations (5.3)-(5.6). This causes the sensitivity of L_0 to β and Z/X_0 to be some 70 per cent greater than is given by Equations (6.2).

7. *The solar neutrino flux*

In principle the solar neutrino luminosity L_ν provides an additional parameter for calibrating solar models. As is well known, that calibration yields an unacceptably low value for the initial helium abundance Y_0 . However, in any discussion of the solar interior it is important to bear in mind the value of L_ν , at least in order to relate it to predictions based on what I might call standard physics (in which, for example, neutrinos are taken to be massless).

According to standard theory, the dominant contribution to the neutrino flux measured by Davis's chlorine detector comes from the decay of ${}^8\text{B}$ in the p-p III chain. I shall therefore discuss only that contribution, $L_{\nu 8}$, to L_ν . If one assumes that evolved theoretical solar models scale homologously under variations of composition, it follows that $L_{\nu 8}$ is simply proportional to the abundance X_8 of ${}^8\text{B}$. By balancing the nuclear reactions of the p-p chain it can easily be shown that the latter is given approximately by

$$X_8 \propto \frac{1-X}{1+X} X^2 \rho T^{24.5} \quad (7.1)$$

(e.g. Gough, 1988). The homology scaling laws of section 4 can now be used to obtain

$$L_{\nu 8} \propto G^{-10.7} M^{-11.8} L^{6.4} (Z/X_0)^{1.8} \propto G^{-13.6} M^{13.8} L^{6.3} Z^{2.0}. \quad (7.2)$$

Once again, I have taken $d = 0.5$ in the opacity formula and have evaluated the exponents for $X_0 = 0.7$. For comparison, partial derivatives quoted by Bahcall and Ulrich (1988) for their standard solar model are

$$\left(\frac{\partial \ln L_{\nu 8}}{\partial \ln L} \right)_{Z/X_0} = 6.8, \quad \left(\frac{\partial \ln L_{\nu 8}}{\partial \ln (Z/X_0)} \right)_L = 1.3. \quad (7.3)$$

The signs of the exponents are easy to understand. First, notice from the proportionalities (4.17) that increasing either X or Z/X decreases L , requiring that any

increase in Z/X requires a compensating decrease in X , given by Equation (5.1), to maintain the current value of L at the observed value L_\odot . An increase in Z/X leads to a greater opacity, despite the increased hydrogen abundance, and consequently a greater temperature gradient in the radiative interior. Therefore T is increased in the core. This is consistent with the requirement from the nuclear reactions that a lower concentration of fuel demands a higher temperature to keep the luminosity at L_\odot . It is evident from the relation (7.1) that $L_{\nu 8}$ is both a decreasing function of X and an increasing function of T ; the dependence of ϱ on X is quite weak and therefore plays only a minor role. Consequently $L_{\nu 8}$ decreases with X and increases with Z/X at constant L .

The dependence of $L_{\nu 8}$ on t cannot be inferred from the simple scaling laws I have been using. The reason is that deviations from homology brought about by the variation of composition in the core increase the central temperature by more than the scaling laws imply, and since $L_{\nu 8}$ is so very sensitive to T (and therefore nearly all the 8B neutrinos are produced in a small region around the centre of the sun), this dominates the variation of $L_{\nu 8}$. Thus, it is perhaps not surprising that if the sun were, say, older than is generally presumed, the increase in $L_{\nu 8}$ caused by the greater inhomogeneity would dominate the compensating decrease due to the increase in X_0 resulting from the solar calibration: $L = L_\odot$ at $t = t_\odot$. Here I simply quote the result from numerical computations reported by Bahcall and Ulrich (1988) for calibrated solar models:

$$\frac{d \ln L_{\nu 8}}{d \ln t_\odot} \simeq 1.3. \quad (7.4)$$

8. *Temporally and spatially varying gravitational constant*

Having discussed the principal aspects of standard main-sequence evolution, we are now in a position to estimate the consequences of relaxing some of the assumptions listed in section 2. I shall discuss two examples explicitly: in this section, that the gravitational constant G is not constant, and in the next section, that the mass M of the star is not constant.

In some cosmologies the gravitational constant decreases with time (using units of mass, length and time in which Planck's constant and the speed of light are constant). Since, according to the relations (4.17) and (4.18), L is a strongly increasing function of G , a declining gravitational constant would imply that main-sequence stars were considerably more luminous in the past than Equations (5.5) or (5.8) imply. The modification that is made can be estimated from an analysis similar to that in section 5.

The variation of G can often be represented by an equation of the form

$$\frac{G(t')}{G(T_u)} = \left(\frac{t'}{T_u}\right)^{-q} \quad (8.1)$$

where T_u is the age of the universe, t' is time measured from the Big Bang and q is a constant. Typically $0 < q \leq 1$. For the evolution of a homogeneous star, Equation (5.1) still holds, but Equation (5.2) must now be replaced by

$$\frac{L}{L_0} = \left(\frac{t'}{T_u}\right)^{-aq} e^{\beta(X-X_0)}, \quad (8.2)$$

where a is given by Equation (4.14). Hence

$$e^{\beta(X-X_0)} \frac{dX}{dt'} = -\frac{L_0}{ME} \left(\frac{t'}{T_u}\right)^{-aq} \quad (8.3)$$

How M varies with t' depends on the cosmology. In most discussions it is constant, and I shall assume that here; I discuss a variation of M separately in the next section. It is then straightforward to integrate Equation (8.3) to determine the variation of X , and thence to substitute the result into Equation (8.2) for the luminosity. If I specialize to the case of the sun, the result may be written

$$\frac{L}{L_\odot} = \left(1 + \frac{t - t_\odot}{T_u}\right)^{-aq} \left\{ 1 - \frac{\lambda T_u}{(aq - 1)\tau_\odot} \left[1 - \left(1 + \frac{t - t_\odot}{T_u}\right)^{1-aq} \right] \right\}^{-1}, \quad (8.4)$$

where

$$\tau_\odot = \frac{EM}{\beta L_\odot} \simeq 1.5 \times 10^{10} \text{y} \quad (8.5)$$

and, as was the case previously, $t = t' - (T_u - t_\odot)$ is time measured from the 'zero-age'

main sequence. In Equation (8.4) I have introduced a factor λ to account for deviations from homology resulting from the inhomogeneity in the core that results from nuclear transmutations. (It is hoped that there will be no confusion with the exponent of ϱ introduced in Eq. (4.2) to represent the opacity). For equation (8.4) to be a solution of Equations (8.2) and (8.3) describing homogeneous evolution, λ must be set to unity; but for the more realistic inhomogeneous case, $\lambda > 1$. It was pointed out at the end of section 5 that standard solar evolution with G constant is well described by Equation (5.5) with τ_n replaced by τ_\odot , which is equivalent to setting $\lambda \cong 1.4$. Thus we should expect Equation (8.4) with $\lambda \cong 1.4$ to provide quite an accurate description of solar evolution when G satisfies Equation (8.1), as indeed it does (c.f. Pochoda and Schwarzschild, 1964; Ezer and Cameron, 1965; Roeder and Demarque, 1966; Shaviv and Bahcall, 1969).

Because L is an increasing function of G , the effect of the variation (8.1) of G is to augment the past luminosity. For values of q and T_u of typical cosmologies, this predominates over the influence of the varying chemical composition represented by Equation (5.8), and L now decreases with time. A very high luminosity in the past would have severe implications for the Earth's climatic history, and can probably be ruled out. However, climatologists have had difficulty reconciling the relatively low past luminosity that is a consequence of standard physics, and it is therefore of interest to ask what values of q and T_u are required to maintain the solar irradiance on Earth at roughly a constant value. In carrying out that calculation one must take into account the influence of G on the radius R_\oplus of the Earth's orbit. Since the timescale $\tau_G = -G/\dot{G}$ (where the dot denotes differentiation with respect to time) is very much greater than the Earth's orbital period, R_\oplus is determined simply by the (Newtonian) orbit equations in which \dot{G} is ignored and angular momentum (the adiabatic invariant) is held constant: thus $R_\oplus \propto G^{-1}$. The solar irradiance on Earth (the solar constant) is thus proportional to $F = [G(t')/G(T_u)]^2 L$. The values of q and T_u required to hold F approximately constant are then such that $\tau_G(T_u) = q^{-1} T_u \cong 1.2 \times 10^{11}$; the precise value is only very weakly dependent on the value of q . An example of $F(t)$, for $q = 1$, is illustrated in Figure 1.

Another possible modification to the theory of gravitation is that the potential V at a distance r from a point mass M is given by

$$V = GM/r \tag{8.6}$$

where

$$G = [1 + \tilde{\alpha} \exp(-r/R_G)] G_\infty, \tag{8.7}$$

where R_G is of order 10^2 m. For $r \ll R_G$ and $r \gg R_G$ the inverse-square law of force applies, but for astronomical distances the gravitational constant is $(1 + \tilde{\alpha})^{-1}$ of the value measured in the laboratory. Since it is GM that is measured astronomically, it follows that the mass of the sun would be $1 + \tilde{\alpha}$ of the usually accepted value. If $\tilde{\alpha}$ and G_∞ are independent of time, the main-sequence evolution must simply follow Equation (5.8), though of course the adjustment of the chemical composition to obtain $L = L_\odot$ at $t = t_\odot$ would be different. The neutrino flux would also be somewhat different; according to the scalings (7.2),

$$L_{\nu 8} \propto (1 + \tilde{\alpha})^{-1.1} (Z/X_0)^{1.8} \propto (1 + \tilde{\alpha})^{-0.2} Z^{2.0} \quad (8.8)$$

at fixed GM and L . This result is to be compared with the numerical computations of Gilliland and Däppen (1987), who find $\partial \ln L_{\nu 8} / \partial \ln(1 + \tilde{\alpha}) \cong -1.2$ at constant Z . For the small value of $\tilde{\alpha}$ suggested by geophysical data, about -7×10^{-3} (Fishbach et al., 1986), the effect on $L_{\nu 8}$ is negligible.

9. Evolution with mass loss

The second example I discuss is the consequence of losing mass, by a wind, during the early stages of main-sequence evolution. This possibility has been modelled recently by Guzik *et al.* (1987), who considered the implications for the sun. Here I consider a mass variation of the form

$$\frac{M}{M_\odot} = \frac{1 + \Lambda e^{-t/t_0}}{1 + \Lambda e^{-t_\odot/t_0}}, \quad (9.1)$$

where Λ and the characteristic mass-loss timescale t_0 are constants. This is similar to the variation considered by Guzik *et al.* for two of their three mass-losing models. Once again Equation (5.1) holds, but now Equation (5.2) is replaced by

$$\frac{L}{L_\odot} = \left(\frac{M}{M_\odot} \right)^e \exp [-\beta(X - X_\odot)], \quad (9.2)$$

where, according to the scaling (4.17), $e = 5.5$. Integrating these equations leads to

$$\frac{L}{L_{\odot}} = \left(\frac{M}{M_{\odot}}\right)^e \left\{1 - \frac{\lambda t_0}{\tau_{\odot}} \left(1 + \Lambda e^{-t_{\odot}/t_0}\right)^{e-1} [I_{e-1}(t/t_0) - I_{e-1}(t_{\odot}/t_0)]\right\}^{-2}, \quad (9.3)$$

where¹

$$I_{\sigma}(x) \equiv \int (1 + \Lambda e^{-x})^{\sigma} dx \quad (9.4)$$

and τ_{\odot} is given by Equation (8.5). I have again introduced the factor λ into the equation for the luminosity to account for the deviations from homology.

As with the case of a declining gravitational constant, mass loss causes the star to be more luminous in the past than is predicted by standard theory, because L is a rapidly increasing function of M . For the short timescales t_0 and the substantial initial mass ($2M_{\odot}$) considered by Guzik *et al.*, there is first a rapid decline in L due to the decline in M , followed by the gentle rise, essentially at constant M , described by Equation (5.8). The initial phases of evolution are not very well described by Equation (9.3), because the star is powered predominantly by the CNO cycle in its high-mass phase. Thus L_0 is overestimated by the formula. However, the formula (9.3) should be a good description for cases in which M_0 is not very much greater than M_{\odot} .

Once again we can ask what parameters are required to maintain the luminosity of the sun at approximately a constant value throughout its main-sequence evolution. This can be carried out by demanding, for example, that $\bar{L}(t_{\odot}) = L_{\odot}$, where

$$\bar{L}(t) = t^{-1} \int_0^t L(t) dt \quad (9.5)$$

is the time-averaged luminosity. One thus determines a relationship between the initial mass $M_0 = M(0)$ and the mass-loss timescale t_0 . Provided t_0/t_{\odot} is not very small, M_0 varies slowly and is such that $L(t)$ is approximately constant. Indeed, as t_0/t_{\odot} increases, M_0 tends to a limit for which $L(0) \simeq L_{\odot}$, which can be estimated from the scaling law (4.17) taking due account of the hydrogen consumed at con-

1. Integrals I_{σ} defined by Equation (9.4) can easily be generated for integral values of σ from the starting value $I_0 = x$ using the recurrence relation $I_{\sigma} = I_{\sigma-1} - \sigma^{-1}(1 + \Lambda e^{-x})^{\sigma}$. I_{σ} is a continuous function of σ , and for the parameters considered here is quite accurately determined for nonintegral values of σ by linear interpolation of $\log I_{\sigma}$ between the integral values. Thus for all the results reported here, $I_{4.5}$ was approximated by the geometric mean of I_4 and I_5 .

stant L in excess of that of the standard model. With the help of Equation (5.5) with τ_n replaced by τ_\odot , this can be written

$$\frac{M_0}{M_\odot} \simeq \left\{ \left(1 + \frac{\lambda t_\odot}{\tau_\odot} \right) \left[1 + \frac{t_\odot}{\beta \tau_\odot X_{0st}} \left(1 - \frac{\bar{L}_{st}(t_\odot)}{L_\odot} \right) \right]^{4.8} \right\}^{1/5.5}, \quad (9.6)$$

$$\simeq 1.08,$$

where X_{0st} is the initial hydrogen abundance of the calibrated standard model and

$$\bar{L}_{st}(t_\odot) = \frac{\tau_\odot L_\odot}{\lambda t_\odot} \ln \left(1 + \frac{\lambda t_\odot}{\tau_\odot} \right) \simeq 0.83 L_\odot \quad (9.7)$$

is the mean main-sequence luminosity of the standard model. When $t_0/t_\odot \ll 1$, the mass of the model quickly approaches M_\odot , and the evolution then follows that given by Equation (5.8). Thus in this case it is not possible to maintain L at an approximately constant value. The constraint $\bar{L}(t_\odot) = L_\odot$ demands a very luminous initial phase, of relatively high mass, as is depicted in Figure 1 for the case $t_0/t_\odot = 0.05$.

10. A further comment on the neutrino flux

Although the precise profile $X(x, t_\odot)$ at the present time must depend on the detailed history of the star, the main factor influencing it is the total amount of hydrogen consumed. This is proportional to the evolutionary age

$$\tau = t/t_{ms} \quad (10.1)$$

where the main-sequence lifetime t_{ms} is a characteristic time taken to deplete hydrogen in the centre of the star². It is evident, therefore, that the principal effect of a decreasing mass or gravitational constant is simply to make the sun look older by roughly a factor \bar{L}/\bar{L}_{st} . (The adjustment due to changes in t_{ms} resulting from recalibrating X_0 is relatively small). According to the dependence (7.4), this increases

2. A convenient precise definition of t_{ms} is twice the time to deplete the central hydrogen abundance to half its initial value.

the 8B neutrino flux by a factor $(\bar{L}/\bar{L}_{st})^{1.3}$, which represents an increase of 30 per cent for the models with $\bar{L} = L_{\odot}$ such as the two represented in Figure 1.

11. *Seismological analysis*

High-order p modes of low degree have been measured from whole-disk Doppler and intensity observations of the sun, and it is likely that similar observations will be successfully carried out on other stars in the near future. The cyclic frequencies ν of oscillations of degree l and order n satisfy for $n \gg l$:

$$\nu \sim \left(n + \frac{1}{2}l + \varepsilon\right) \nu_0 - \frac{A l(l+1) - B}{n + \frac{1}{2}l + \varepsilon} \nu_0 + \dots, \quad (11.1)$$

where

$$\nu_0 = \left(2 \int_0^R \frac{dr}{c}\right)^{-1} \quad (11.2)$$

and

$$A = \frac{1}{4\pi^2\nu_0} \left[\frac{c(R)}{R} - \int_{r_t}^R \frac{1}{r} \frac{dc}{dr} dr \right] \quad (11.3)$$

(e.g. Gough, 1986). Here $c(r)$ is the adiabatic sound speed in the star (rather than the speed of light), and ε and B are constants of the model that depend mainly on conditions in the very outer layers. (Note that ε introduced here is not to be confused with the nuclear energy generation rate.) The lower limit of integration r_t in Equation (11.3) is the radius of the inner turning point of the oscillation eigenfunction, and is given by

$$c(r_t)/r_t = 2\pi\nu/l. \quad (11.4)$$

The upper limit, R , is to be interpreted as the radius at which $\nu = \nu_c$, where

$$\nu_c = \frac{c}{2\pi H} \left(1 - 2 \frac{dH}{dr} \right)^{\frac{1}{2}}, \quad (11.5)$$

H being the density scale height; for the high-order modes considered here, R is only very slightly less than the radius of the photosphere. Expressions (11.1)-(11.3) with $r_t = 0$ and R equal to the radius of the star can be deduced from the asymptotic analysis of Tassoul (1980), and as $n/l \rightarrow \infty$ this is formally equivalent to taking r_t and R as the turning-point radii.

Only modes with $l \leq 3$ are likely to be observed in stars other than the sun, at least initially, and from these can be constructed the quantities

$$\Delta_{n\ell} = \nu_{n,\ell} - \nu_{n-1,\ell} \simeq \left[1 + \frac{A\ell(\ell+1) - B}{\left(n + \frac{1}{2}\ell + \varepsilon\right)^2} \right] \nu_0 \simeq \nu_0, \quad (11.6)$$

$$\begin{aligned} d_{n\ell} &= \frac{3}{2\ell+3} (\nu_{n,\ell} - \nu_{n-1,\ell+2}) \simeq \frac{6A\nu_0}{n + \frac{1}{2}\ell + \varepsilon} \\ &\simeq -\frac{3}{2\pi^2 \left(n + \frac{1}{2}\ell + \varepsilon\right)} \int_{r_t}^R \frac{1}{r} \frac{dc}{dr} dr \end{aligned} \quad (11.7)$$

and

$$\Phi_{n\ell} = \frac{\nu}{\nu_0} - \left(n + \frac{1}{2}\ell\right) \nu_0 \simeq \varepsilon + \frac{B}{n + \frac{1}{2}\ell + \varepsilon} - \frac{\ell(\ell+1)d_{n\ell}}{6\nu_0}, \quad (11.8)$$

where, for simplicity, I have neglected in Equation (11.7) the small first term in square brackets in the expression (11.3) for A , and I have also ignored the l dependence of r_t . The quantity $\Delta_{n\ell}$ is an integral property of the entire star, $d_{n\ell}$ depends predominantly on conditions in the core (since it is an integral of the sound-speed gradient weighted by r^{-1}), and $\varepsilon + B/(n + \frac{1}{2}l + \varepsilon)$ is a functional of the stratification principally in the surface layers.

I must emphasize that the asymptotic relations (11.6)-(11.8) are not accurate. In particular, the $(n + \frac{1}{2}l + \varepsilon)^{-1}$ dependence of the small frequency separation $d_{n\ell}$ is not well satisfied by either numerically computed eigenfrequencies or the solar frequen-

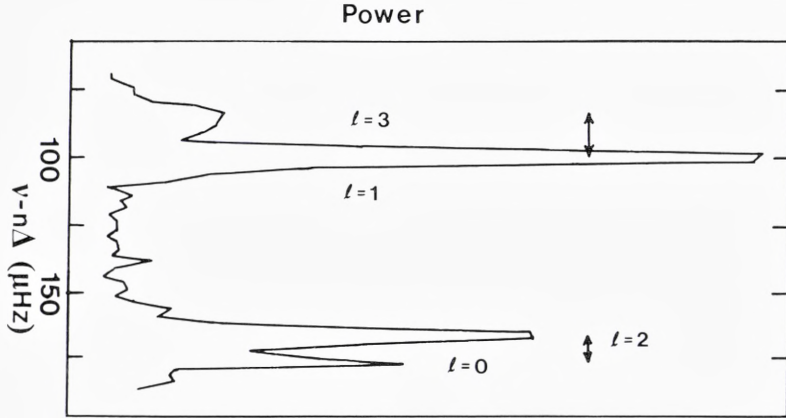


Fig. 2. Superposed power spectrum of solar whole-disk Doppler data obtained by Grec *et al.* (1980). It is a (somewhat optimistic) representation of what might soon be available from observations of other stars. The diagram is obtained by dividing the frequency axis into intervals (ν_{n-1}, ν_n) , where $\nu_n = 136n$ μHz , and summing the power. The dissection interval (136 μHz) was chosen to maximize some measure of overlap of the peaks, and determines Δ . The double-headed arrows represent $d_0 \approx 10 \mu\text{Hz}$ and $(5/3)d_1 \approx 17 \mu\text{Hz}$; the location of the $l = 0$ peak (177 μHz) determines Φ (≈ 1.30).

cies; this inaccuracy is due partly to the neglect of buoyancy and the perturbation of the gravitational potential in the asymptotic analysis, both of which are strongly l dependent. Nevertheless, Equations (11.6) and (11.7) do give some idea of the dominant term that determines the dependence of the measured quantities on the structure of the star, and I shall use them in this discussion as a guide to understanding more accurate quantitative comparisons.

The first stellar observations to reveal a profusion of high-order p modes are unlikely to resolve modes of different degree with like $n + \frac{1}{2}l$; the frequency difference d_{nl} is only of order 10 μHz . However, since groups of modes with like $n + \frac{1}{2}$ are approximately uniformly spaced in frequency, a superposed frequency analysis of the kind first carried out by Grec *et al.* (1980) for the sun (illustrated in Figure 2) is likely to yield average values Δ , d and Φ of Δ_{nl} , d_{nl} and Φ_{nl} , from which ν_0 , A and a measure of the surface conditions can be estimated. Since the physics of the outer layers of stars is very complicated and ill-understood, Φ_{nl} has not been satisfactorily reproduced theoretically even for the sun. I shall therefore confine my discussion to Δ and d .

12. The solar calibration

From the discussion in section 6 it is evident that standard solar models form a one-parameter sequence, which can be labelled by, say, the initial helium abundance Y_0 . This assumes, of course, that the age t_\odot of the sun since the zero-age main sequence is known. Knowledge of Δ or d therefore affords a basis for selecting the most representative model. The first attempt to calibrate Y_0 , using both Δ and d , failed to produce consistent results (Christensen-Dalsgaard and Gough, 1980): a

Source	$d_0(\mu\text{Hz})$	$d_1(\mu\text{Hz})$
Observations	9.2 ± 0.6	9.7
Standard solar models	10.3 ± 0.3	10.0 ± 0.6
Diffusively mixed models	15.5 ± 2.0	13.6 ± 2.3
Helium-deficient model	13	11
Model with varying G	9.0	9.2
Model with mass loss	9.0	9.2
Wimp-infested model	9.3 ± 0.3	9.0 ± 0.3

Table 1. Mean normalized solar p-mode frequency separations d_i , which are averages over n of the quantities d_{ni} defined by Equation (11.7). The observations are averages of d_i from Claverie *et al.* (1981), Grec *et al.* (1983), Woodard and Hudson (1983) and Harvey and Duvall (1984) \pm one standard deviation of the results quoted, taking no account of the (generally smaller) estimated observational errors. The separations for standard solar models are averages of the results of Christensen-Dalsgaard (1982) and Ulrich and Rhodes (1984) \pm one standard deviation of those results taking no account of computational inaccuracy. The values of d_i for diffusively mixed models are averages of the results taken from Ulrich and Rhodes (1983), Berthomieu *et al.* (1984), Christensen-Dalsgaard (1986) and Cox and Kidman (1984); the standard deviations are higher here because the authors did not all make the same assumptions about how material was mixed. The entries for the model with wimps were taken from the estimates by Faulkner *et al.* (1986) and Däppen *et al.* (1986) of frequencies of a model by Gilliland *et al.* (1986). The helium-deficient model is that with $Y_0 = 0.19$ computed by Christensen-Dalsgaard, Gough and Morgan (1979). To provide a meaningful comparison the theoretical separations quoted here were computed by averaging the differences from the standard models obtained by each author and adding the result to the mean values quoted in the second row of the table. The models with decreasing G and decreasing M are those illustrated in Figure 1, for which $\bar{L}(t_\odot) = L_\odot$.

calibration based on Δ yielded a low value, about 0.19, for Y_0 whereas, as can easily be deduced from the entries in Table 1, a calibration based on d yields $Y_0 \approx 0.27$. Since d depends primarily on conditions in the core where the chemical composition has been modified by nuclear reactions, whereas Δ depends on conditions throughout the star including the uncertain outer layers, one might at first sight put more trust in the value of d , despite the fact that the low value of Y_0 based on Δ

yields a value for the neutrino flux which is not significantly at variance with observation. In the case of the sun, this opinion was upheld by a calibration using high-degree modes (Berthomieu *et al.*, 1980), but I shall not discuss that here since such modes are not going to be observed in other stars in the foreseeable future. Suffice it to say that a recent computation by Christensen-Dalsgaard *et al.* (1988) using an improved equation of state, which differs from previous equations of state mainly in the upper layers of the convective envelope, has diminished the inconsistency substantially, and thus suggests that the higher value of Y_0 is more-or-less correct.

It is important to realise that the standard view of main-sequence stellar evolution may not be correct, or that the generally accepted value of t_\odot may be in error. So far as the influence on d is concerned, the major factor is the profile of mean molecular weight $\mu(r,t)$, which influences the sound speed c . Let us assume the perfect-gas law, so that

$$c^2 = \gamma \frac{\mathcal{R}T}{\mu}, \quad (12.1)$$

where γ is the adiabatic exponent $(\partial \ln p / \partial \ln \varrho)_s$, the derivative being taken at constant specific entropy s . Then since the dependence of the nuclear energy generation rate on T is quite sensitive, the gross structure of the star tends to constrain the variation of T quite severely, leaving μ to have the major influence on c .

We are now in a position to make a rough estimate of the variation of d with time. By estimating the variation $X(r,t)$ from the hydrogen depletion by nuclear reactions, and considering the outcome to be but a small perturbation from an homologous evolution, the variation of d for late-type stars (with radiative interiors) is given approximately by

$$d(\tau) \simeq (1 - \Psi\tau) \tilde{d}, \quad (12.2)$$

where Ψ is a constant whose value (about 0.7 for sun-like stars) can easily be estimated from numerically computed stellar models, and \tilde{d} is the value of d for a homogeneous stellar model of the same mass and radius. Thus one might expect \tilde{d} to scale approximately according to the homology relation

$$\tilde{d} \propto M^{\frac{1}{2}} R^{-\frac{3}{2}}, \quad (12.3)$$

where, since \tilde{d} is determined mainly by conditions in the radiative interior, R should be expected to characterize the interior scale rather than the entire star. The separation Δ , on the other hand, which depends more sensitively on the outer regions of the star where c is relatively low, is not substantially affected by the details of the μ profile in the core, and would be expected to scale as

$$\Delta \propto M^{\frac{1}{2}} R^{-\frac{3}{2}}, \quad (12.4)$$

where R is the stellar radius. In practice, the relation (12.4) is quite well satisfied, whereas the relation (12.3) is best approximated with a value of R that is intermediate between the stellar radius and that given by the homology law (4.11) which is presumed to represent the scale of the radiative interior. The appropriate R increases as the star evolves, and therefore d decreases with time. If one ignores the difference in the meanings of R in Equations (12.3) and (12.4), it is evident that the nonhomologous component of the evolution is characterized by

$$\delta = d/\Delta \simeq 1 - \Psi\tau. \quad (12.5)$$

As Ulrich (1986) has pointed out, this is a more direct measure of the evolutionary age.

As I mentioned earlier, the asymptotic formulae for the frequency separations Δ and d are not accurate enough for computing reliable theoretical values for comparison with observation. However, they are adequate, and indeed often very useful, for a first estimate of the frequency change produced by some variation to a stellar model. For example, as shown in Table 1, the value of d computed from the standard model 1 of Christensen-Dalsgaard (1982) is about $0.9\mu\text{Hz}$ higher than the mean of the solar observations.³ It is clear from Table 1 that at the present level of reliability of both the theory and the observations the discrepancy is not significant. Nevertheless, it is evident that that discrepancy could actually be removed if the sun were somewhat older, by an amount

$$\delta t \simeq -\frac{0.9t_{\odot}}{d} \left(\frac{d \ln d}{d \ln t} \right)^{-1}, \quad (12.6)$$

3. Exactly how the averaging of d_{nl} is carried out to obtain d is not important for this discussion; all that is important is that the averaging be carried out in the same way for all data sets to be compared.

where d is measured in μHz . In making this estimate I ignored the contribution arising from the change in X required to maintain the observed luminosity; that is very small, as can easily be deduced from Equation (5.8) and the scaling (4.17). Thus from Equation (12.2), the scaling law (12.3) with R being the solar radius, and the temporal derivative of R quoted in Table 2, one obtains $\delta t \cong 6 \times 10^8$ y. Note, however, that gravitational settling of helium would also reduce d by increasing the μ gradient, but I make no attempt here to estimate the magnitude of the effect. Settling of heavier elements increases the opacity in the core; this also increases the central condensation and hence reduces d , though the effect is likely to be less than a comparable settling of helium.

It is evident that we can now immediately estimate the value of d for the models with varying G or M discussed in sections 8 and 9. As was recognized in section 10, it is the total amount of hydrogen consumed that is the principal determinant of the structure of an evolved main-sequence star, so that d can be estimated to be approximately the same as that of a standard model of age $(\bar{L}/\bar{L}_{\text{st}})t_{\odot}$. Hence, for the two models with $\bar{L} = L_{\odot}$ depicted in Figure 1, the age of the equivalent standard model would be $t_{\odot}/0.83$, which is about 9×10^8 y greater than t_{\odot} . Hence d is about $1.4\mu\text{Hz}$ less than the standard value, and is currently within the limits set by the observations. According to this analysis, the two models satisfying Equation (9.1) that Guzik *et al.* (1987) considered, with $t_0 = 1.3 \times 10^8$ y and 3.3×10^8 y and each with $M_0 = 2M_{\odot}$, have present values $d \cong 8\mu\text{Hz}$ and $6\mu\text{Hz}$ respectively; these values are somewhat lower than the values $9.5\mu\text{Hz}$ and $8.5\mu\text{Hz}$ computed recently by Turck-Chièze, Däppen and Casse (1988). The value of d for a model for which Equations (8.6) and (8.8) hold is influenced predominantly by the modification to the initial hydrogen abundance required for the solar calibration. Thus

$$\frac{\partial \ln d}{\partial \ln (1 + \tilde{\alpha})} \simeq \frac{\partial \ln d}{\partial \ln X_0} \frac{\partial \ln X_0}{\partial \ln (1 + \tilde{\alpha})}, \quad (12.7)$$

the partial derivatives being taken at constant GM and Z/X_0 . Using the values of the derivations in Table 2 one thus deduces a decrease in d below that of the standard model of $-1.1\tilde{\alpha}d \cong 0.08\mu\text{Hz}$, if $\tilde{\alpha} = -7 \times 10^{-3}$. This is consistent with the very small values found numerically by Gilliland and Däppen (1987).

I must point out that these simple scaling laws do not always give the correct result. For example, one might try to estimate the value of d for a helium-deficient solar model. Since, as was pointed out above, the nonhomologous contribution to d due to composition changes is small, two calibrated models with the same value of τ have essentially the same value of d . Evidently $t_{\text{ms}} \propto X_0$ for models calibrated to have the correct luminosity, so $\delta\tau/\tau = -\delta X_0/X_0$ at fixed L . In other words, a

model with a value of Y_0 that is, say, 0.08 smaller than the standard appears to be younger by $0.08X_0^{-1}t_\odot \cong 5 \times 10^8$ y. Thus we would expect d to be about $1.2\mu\text{Hz}$ higher. As can be seen in Table 1, this is rather greater than the numerical calculations of d_1 (d_1 is an average over n of d_{nl}), but substantially less than the computed values of d_0 . The reason is presumably that neither the $l = 3$ nor the $l = 1$ modes, which determine d_1 , penetrate to the very centre of the star, whereas the $l = 0$ modes do. So perhaps there is a severe nonhomologous influence very close to $r = 0$.

Similar remarks apply to diffusively mixed models. The most extreme of a completely mixed model would be a zero-age $1M_\odot$ star calibrated to the solar radius and luminosity, for which we would expect $d \approx 13\mu\text{Hz}$. The scatter amongst the various numerical computations is too great to make a sound comparison, though the estimate given here appears to be somewhat too low.

Finally I include in Table 1 the results of an estimate made by Faulkner *et al.* (1986) from the asymptotic formula (11.7) of a solar model with a cloud of weakly interacting massive particles (wimps) in its interior. The effect of wimps is to introduce an additional energy-transporting agent into the inner regions, thereby making the core more nearly isothermal. Despite the resulting reduction in the μ gradient, the effect is to reduce the sound speed preferentially near the centre, and so decrease d . Numerical calculations by Däppen *et al.* (1986) have confirmed this result. The entries in Table 1 are averages of the two calculations.

13. Asteroseismological calibration

For stars of known chemical composition, the seismological diagnostic quantities Δ and d provide a very important supplement to the usual astronomical data. In particular, for zero-age main-sequence stars, the scaling (12.4), together with a mass-radius relation, should in principle permit one to infer the mass of a star. Note that if one ignores the difference between the meanings of R in the scalings (12.3) and (12.4), a knowledge of d would at first sight provide no new information. However, if that difference is taken into account, one would expect Δ and d to scale rather differently with chemical composition, and therefore some overall abundance information would also be available.

For older stars there is the added richness afforded by the temporal evolution. Since both d and Δ change with time, one might expect to be able to determine both the mass and the age of a star. This was first pointed out by Christensen-Dalsgaard (1986), but would evidently be the case only if all the other uncertain parameters determining the structure of the stellar model were known (and, of course, provided that the implementation of the theory of stellar evolution were correct).

Before addressing the issue of what might actually be learned from Δ and d to constrain the possible structure of a stellar model, it is useful first to consider how Δ and d depend on M and t . To this end it is necessary to determine how R varies with time. Since, according to the analysis of section 5, the evolution results solely from the depletion of X which causes an augmentation of L , I shall adopt the approximation $\partial \ln R / \partial \ln \tau = (\partial \ln R / \partial \ln L) \partial \ln L / \partial \ln \tau$, where the partial derivative $(\partial \ln R / \partial \ln L)$ is at constant M , τ , Z and α , and can thus be deduced from the scalings (4.17) and (4.19), and the time derivatives represent the evolution of a given model (at constant M , Z and α). Thus $\partial \ln L / \partial \ln \tau$ is obtained by differentiating Equation (5.8). The value of this derivative, together with the other partial derivatives of R and L obtained from the scalings (4.17) and (4.19) and Equation (5.8) are listed in the first two columns of Table 2. From the derivative of Equation (12.2) and the scalings (12.3) and (12.4), the partial derivatives of Δ and d can thus be determined. These too are listed in Table 2. (For this purpose the scaling

a_i/A_k	$\ln R$	$\ln L$	$\ln T_e$	$\ln \Delta$	$\ln d$
$\ln X_0$	-1	-4.8	-0.7	1.5	1.5
$\ln Z$	-0.5	-0.55	0.11	0.75	0.75
$\ln M$	1.2	5.5	0.78	-1.3	-1.3
$\ln \alpha$	-0.2	0	0.10	0.3	0.3
$\ln \tau$	0.09	0.4	0.06	-0.14	-0.68

Table 2. Partial derivatives $\partial A_k / \partial a_i$ of the properties $A_k = (\ln R, \ln L, \ln T_e, \ln \Delta, \ln d)$ of a standard solar model with respect to the control variables $a_i = (\ln X_0, \ln Z, \ln M, \ln \alpha, \ln \tau)$.

of \tilde{d} has been assumed to be the same as that of Δ .) It is evident that both Δ and d decrease with time, d decreasing roughly five times faster than Δ . The results are plotted against each other in Figure 3, which, in view of the cavalier way in which I have simplified the analysis, can be regarded as only a very crude representation of what actually occurs. The careful numerical computations reported by Christensen-Dalsgaard (1988), however, are at least superficially similar.

Knowledge of Δ and d fixes a point in Figure 3, to which correspond values of M and τ (and hence t). But that is unambiguously the case only if the chemical abundance parameters, X_0 and Z , and the mixing-length parameter α are known, together with, say, the luminosity L . How well M and t can be determined depends on the errors in the other astronomical information about the star and on the sensitivity of the asteroseismological analysis to those errors. This issue has been partially addressed by Ulrich (1986, 1988), who determined the information required to obtain the entries in Table 2 from numerical computations. His results differ

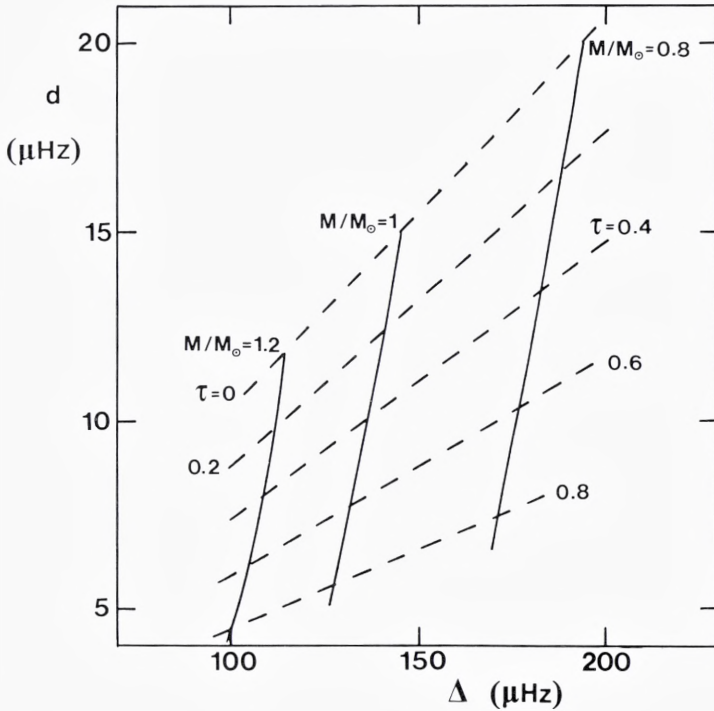


Fig. 3. Dependence of d upon Δ for late-type main-sequence stars, estimated from the rough scaling arguments described in the text. The solid lines represent the evolution of a given star ($M = \text{constant}$); the dashed lines are evolutionary isochrones (lines of constant $\tau = t/t_{ms}$). If δ rather than d were plotted against Δ , the evolutionary isochrones, in this approximation, would be uniformly spaced horizontal lines.

somewhat from the results I have obtained from the simple scaling laws, but that will not alter the qualitative nature of my principal conclusion.

Let $a_i = (\ln X_0, \ln Z, \ln \alpha, \ln M, \ln \tau)$ be the control parameters for the theoretical stellar model, and let $A_k = (\ln T_e, \ln L, \ln \Delta, \ln d)$ be the output. Then from the partial derivatives $\partial A_k / \partial a_i$ of Table 2 one knows that small changes δa_i in the control parameters produce changes δA_k in the output, according to the linearized equations

$$\frac{\partial A_k}{\partial a_i} \delta a_i - \delta A_k = 0. \tag{13.1}$$

One can now ask how any four parameters B_k , say, selected from the nine parameters a_i , A_k , depend on the remaining parameters b_i . More generally, the nine parameters b_i , B_k can be any set of independent functions of a_i , A_k . The appropriate relations between small perturbations δb_i , δB_k to b_i , B_k are, of course, simply Equations (13.1) rewritten in terms of the new variables thus:

$$\left(\frac{\partial A_k}{\partial a_i} \frac{\partial a_i}{\partial B_m} + \frac{\partial A_k}{\partial B_m} \right) \delta B_m + \left(\frac{\partial A_k}{\partial a_i} \frac{\partial a_i}{\partial b_j} + \frac{\partial A_k}{\partial b_j} \right) \delta b_j = 0, \quad (13.2)$$

which can be solved formally for δB_m in terms of δb_j . If one then sets $b_j = \delta_{ji}$ (where δ_{ij} is the Kronecker delta), to obtain the solution $\delta_i B_k$, say, the appropriate sensitivity matrix is given by

$$\frac{\delta_i B_k}{\delta b_j} = \frac{\partial B_k}{\partial b_i}. \quad (13.3)$$

In Equation (13.2) the partial derivatives $\partial A_k / \partial a_i$ are defined regarding A_k to be functions of a_i as in Equation (13.1); thus they are the partial derivatives given in Table 2. The derivatives $\partial \alpha_i / \partial \beta_j$, where α_i is any of a_i or A_k and β_j is b_i or B_k , are the partial derivatives of the transformation $\alpha_i = \alpha_i(\beta_j)$ between the two sets of variables. The resulting partial derivatives in equation (13.3) are thus taken with the components B_k considered as functions b_i , analogously to those of Equation (13.1).

The results of such a transformation is given in Table 3, where I have taken $B_k = (\ln t, \ln M, \ln Y, \ln \alpha)$ in the hope that they can be determined in terms of $b_i = (200 \ln T_e, 10 \ln L, 10 \ln(Z/X_0), \Delta, d)$, where Δ and d are measured in μHz . In so

b_i	δb_i	$\delta t/t$	$\delta M/M$	$\delta Y/Y$	$\delta \alpha/\alpha$
T_e	$0.005 T_e$	0.003	-0.03	0.09	0.13
L	$0.1 L$	-0.012	0.14	-0.38	-0.49
Z/X_0	$0.1 Z/X_0$	-0.078	-0.17	1.8	-7.4
Δ	$1 \mu\text{Hz}$	0.013	0.01	-0.05	-0.04
d	$1 \mu\text{Hz}$	-0.19	-0.01	0.08	0.08

Table 3. Sensitivity matrix $\partial B_k / \partial b_i$. The entries in columns 3-6 represent the contributions to the relative errors in calibrating t , M , Y or α from the errors δb_i in b_i that are listed in column 2.

doing, I have assumed that $t_{ms} \propto X_0/L$. The factors in the definition of b_i have been chosen such that a value of unity for any component δb_i represents a (perhaps optimistic) estimate of the smallest error one might make from the observation of an isolated sun-like star. Thus the entries in Table 3 represent the contributions from those errors in b_i to the resulting relative errors in t , M , Y and α . It is evident from the magnitude of the entries that from a knowledge of the position of a star on Figure 3 the mass and age of that star can each be determined to within about 20%. However one cannot determine the initial helium abundance Y_0 . A somewhat less optimistic conclusion is reached (Gough, 1987) if one replaces Table 2 by the numerical derivatives one deduces from the data provided by Ulrich (1986, 1988)⁴.

The relatively high sensitivity of the components of B_k of Table 3 to Z/X_0 suggests that perhaps Z/X_0 should not be used as a control variable. This conclusion is strengthened by the realization that Z enters the equations principally via the opacity, and that therefore this analysis also reflects the sensitivity of the results to errors in the uncertain results of opacity calculations. Thus, if surface gravity, for example, were used instead, or if a determination of M could be made for a component of a binary system, then perhaps Z/X_0 and τ could be determined more reliably (Gough, 1987).

The ramifications of this seismological sensitivity analysis, even for sun-like stars, have not yet been fully explored. Nor has the analysis been carried out for stars whose structure differs substantially from that of the sun. Therefore it is not yet possible to make a quantitative assessment of the role the anticipated asteroseismic data will play. It is already evident from the difference between Table 3 and an analogous table (Gough, 1987) computed from presumably more accurate numerical calculations by Ulrich that variations in the values in Table 2 can produce quite substantial changes in the outcome of the analysis. Thus there might be regions in Figure 3 that are significantly less sensitive to the uncertain control parameters.

14. Conclusion

The purpose of my introductory discussion has been mainly to provide some insight into the structure and evolution of sun-like stars on the main sequence. I have avoided use of the results of complicated numerical calculations as much as possible, relying on simple scaling arguments wherever I could. This is in character with many

4. My earlier conclusion (Gough, 1987) was very much less optimistic than that arrived at here. The main reason is that errors in modern spectroscopic determinations of T_e and Z/X_0 for stars observed with the precision required for detecting a spectrum of p-mode oscillations (cf. Smith et al., 1986; Perrin et al., 1988; Spite et al., 1989) are substantially less than the older estimates which I had used.

of the arguments Strömgren himself advanced during his very productive career. Although the results of scaling arguments are not as accurate as detailed numerical calculations, and should not be used for quantitative comparisons between theory and observation if precise numerical results are available, they do provide a means by which some understanding of numerical computations can be achieved, and they are also often a ready tool for making preliminary estimates of the effects of some small change to the assumptions of the theory. Thus they are an extremely important complement to numerical modelling.

I have not attempted a serious review of the literature, but have instead concentrated on physical rationalizations. Much of what I have discussed is very well known, but I have also extended the technique to address some of the seismic properties of stars, which are of only quite recent interest and which promise to become an extremely important diagnostic of stellar structure. Thus I hope that this contribution will be of some practical scientific use.

One of the issues to which Strömgren devoted considerable attention is the helium content of main-sequence stars. With the addition of anticipated new asteroseismic data I have readdressed that subject, originally in the hope that substantial progress might soon be made. The initial results were perhaps disappointing, because the inferred helium abundance appears to be extremely sensitive to uncertainties in the astronomical data. However, sensitivity in one direction always implies that there is a direction, in some sense opposite, in which there is very little sensitivity; and that is perhaps the direction to go to make reliable physical deductions. Therefore we can look forward to the new data in the optimistic belief that important new constraints will be set on the possible structure of main-sequence stars, which no doubt will lead to an improvement in our understanding of stellar evolution.

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I am very grateful to P. E. Nissen for pointing out that the work of Smith *et al.* (1986) and Perrin *et al.* (1988) shows that careful spectroscopic analysis can yield values of T_e and Z/X with only 0.5% and 10% errors respectively, and to P. S. Conti for drawing my attention to the work of Spite *et al.* (1989). I thank C. A. Morrow and E. Novotny for suggesting improvements to the original manuscript.

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